Published by Institute of Physics Publishing for SISSA/ISAS
Received: *April 8, 2002*
Accepted: *April 8, 2002*

Strings in flat space and pp waves from $\mathcal{N} = 4$ super Yang Mills

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Abstract: We explain how the string spectrum in flat space and pp-waves arises from the large-N limit, at fixed $g^2_{\rm YM}$, of U(N) $\mathcal{N} = 4$ super Yang Mills. We reproduce the spectrum by summing a subset of the planar Feynman diagrams. We give a heuristic argument for why we can neglect other diagrams.
We also discuss some other aspects of pp-waves and we present a matrix model associated to the DLCQ description of the maximally supersymmetric eleven dimensional pp-waves.

Keywords: 1/N Expansion, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

The fact that large-N gauge theories have a string theory description was believed for a long time [1]. These strings live in more than four dimensions [2]. One of the surprising aspects of the AdS/CFT correspondence [3]–[6] is the fact that for $\mathcal{N} = 4$ super Yang Mills these strings move in ten dimensions and are the usual strings of type IIB string theory. The radius of curvature of the ten dimensional space goes as $R/l_s \sim (g_{\rm YM}^2 N)^{1/4}$. The spectrum of strings on $AdS_5 \times S^5$ corresponds to the spectrum of single trace operators in the Yang Mills theory. The perturbative string spectrum is not known exactly for general values of the 't Hooft coupling, but it is certainly known for large values of the 't Hooft coupling where we have the string spectrum in flat space. In this paper we will explain how to reproduce this spectrum from the gauge theory point of view. In fact we will be able to do slightly better than reproducing the flat space spectrum. We will reproduce the spectrum on a pp-wave. These pp-waves incorporate, in a precise sense, the first correction to the flat space result for certain states.

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The basic idea is the following. We consider chiral primary operators such as $\mathrm{Tr}[Z^J]$ with large J. This state corresponds to a graviton with large momentum p^+. Then we consider replacing some of the Zs in this operator by other fields, such as ϕ, one of the other transverse scalars. The position of ϕ inside the operator will matter since we are in the planar limit. When we include interactions ϕ can start shifting position inside the operator. This motion of ϕ among the Zs is described by a field in 1+1 dimensions. We then identify this field with the field corresponding to one of the transverse scalars of a string in light cone gauge. This can be shown by summing a subset of the Yang Mills Feynman diagrams. We will present a heuristic argument for why other diagrams are not important.

Since these results amount to a "derivation" of the string spectrum at large 't Hooft coupling from the gauge theory, it is quite plausible that by thinking along the lines sketched in this paper one could find the string theory for other cases, most interestingly cases where the string dual is not known (such as pure non-supersymmetric Yang Mills).

We will also describe other aspects of the physics of plane waves. For example we consider the M-theory plane wave background with maximal supersymmetry [11, 12] and we show that there is an interesting matrix model describing its DLCQ compactification. This matrix model has some unusual features such as the absence of flat directions. We merely touch the surface on this topic in section 5, postponing a more detailed investigation for the future.

This paper is organized as follows. In section 2 we will describe a limit of $AdS_5 \times S^5$ that gives a plane wave. In section 3 we describe the spectrum of string theory on a plane wave. In section 4 we describe the computation of the spectrum from the $\mathcal{N} = 4$ Yang Mills point of view. In section 5 we describe the Matrix model associated to the DLCQ compactification of the M-theory plane wave and discuss some of its features. In appendix A we describe in detail some of the computations necessary for section 4. In appendix B we prove the supersymmetry of the Matrix model of section 5. In appendix C we describe the string spectrum on a plane wave with mixed NS and RR backgrounds.

2. pp waves as limits of $AdS \times S$

In this section we show how pp wave geometries arise as a limit of $AdS_p \times S^q$ [1]. Let us first consider the case of $AdS_5 \times S^5$. The idea is to consider the trajectory of a particle that is moving very fast along the S^5 and to focus on the geometry that this particle sees. We start with the $AdS_5 \times S^5$ metric written as

$$ds^2 = R^2 \left[-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\psi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega'^2_3 \right]. \qquad (2.1)$$

We want to consider a particle moving along the ψ direction and sitting at $\rho = 0$ and $\theta = 0$. We will focus on the geometry near this trajectory. We can do this systematically

[1] While this paper was being written the paper [26] appeared which contains the same point as this section.

by introducing coordinates $\tilde{x}^{\pm} = \frac{t \pm \psi}{2}$ and then performing the rescaling

$$x^+ = \tilde{x}^+ \,, \qquad x^- = R^2 \tilde{x}^- \,, \qquad \rho = \frac{r}{R} \,, \qquad \theta = \frac{y}{R} \,, \qquad R \to \infty \,. \tag{2.2}$$

In this limit the metric (2.1) becomes

$$ds^2 = -4dx^+ dx^- - (\vec{r}^{\;2} + \vec{y}^{\;2})(dx^+)^2 + d\vec{y}^{\;2} + d\vec{r}^{\;2} \,, \tag{2.3}$$

where $\vec{y}$ and $\vec{r}$ parametrize points on R^4. We can also see that only the components of F with a plus index survive the limit. We see that this metric is of the form of a plane wave metric[2]

$$\begin{aligned} ds^2 &= -4dx^+ dx^- - \mu^2 \vec{z}^{\;2} dx^{+2} + d\vec{z}^{\;2} \\ F_{+1234} &= F_{+5678} = \text{const} \times \mu \,, \end{aligned} \tag{2.4}$$

where $\vec{z}$ parametrizes a point in R^8. The mass parameter μ can be introduced by rescaling (2.2) $x^- \to x^-/\mu$ and $x^+ \to \mu x^+$. These solutions where studied in [7].

It will be convenient for us to understand how the energy and angular momentum along ψ scale in the limit (2.2). The energy in global coordinates in AdS is given by $E = i\partial_t$ and the angular momentum by $J = -i\partial_\psi$. This angular momentum generator can be thought of as the generator that rotates the 12 plane of R^6. In terms of the dual CFT these are the energy and R-charge of a state of the field theory on $S^3 \times R$ where the S^3 has unit radius. Alternatively, we can say that $E = \Delta$ is the conformal dimension of an operator on R^4. We find that

$$\begin{aligned} 2p^- &= -p_+ = i\partial_{x^+} = i\partial_{\tilde{x}^+} = i(\partial_t + \partial_\psi) = \Delta - J \\ 2p^+ &= -p_- = -\frac{\tilde{p}_-}{R^2} = \frac{1}{R^2} i\partial_{\tilde{x}^-} = \frac{1}{R^2} i(\partial_t - \partial_\psi) = \frac{\Delta + J}{R^2} \,. \end{aligned} \tag{2.5}$$

Notice that $p^{\pm}$ are non-negative due to the BPS condition $\Delta \geq |J|$. Configurations with fixed non zero p^+ in the limit (2.2) correspond to states in AdS with large angular momentum $J \sim R^2 \sim N^{1/2}$. When we perform the rescalings (2.2) we take the $N \to \infty$ limit keeping the string coupling g fixed and we focus on operators with $J \sim N^{1/2}$ and $\Delta - J$ fixed.

From this point of view it is clear that the full supersymmetry algebra of the metric (2.1) is a contraction of that of $AdS_5 \times S^5$ [7]. This algebra implies that $p^{\pm} \geq 0$.

This limit is a particular case of Penrose's limit [8],[3] see also [9, 10]. In other $AdS_d \times S^p$ geometries we can take similar limits. The only minor difference as compared to the above computation is that in general the radius of AdS_d and the sphere are not the same. Performing the limit for $AdS_7 \times S^4$ or $AdS_4 \times S^7$ we get the same geometry, the maximally supersymmetric plane wave metric discussed in [11, 12]. For the $AdS_3 \times S^3$ geometries that arise in the D1-D5 system the two radii are equal and the computation is identical to the one we did above for $AdS_5 \times S^5$.

[2]The constant in front of F depends on the normalizations of F and can be computed once a normalization is chosen.

[3]We thank G. Horowitz for suggesting that plane waves could be obtained this way.

In general the geometry could depend on other parameters besides the radius parameter R. It is clear that in such cases we could also define other interesting limits by rescaling these other parameters as well. For example one could consider the geometry that arises by considering D3-branes on A_{k-1} singularities [13]. These correspond to geometries of the form $AdS_5 \times S^5/\mathbb{Z}_k$ [14]. The $\mathbb{Z}_k$ quotient leaves an S^1 fixed in the S^5 if we parametrize this S^1 by the ψ direction and we perform the above scaling limit we find the same geometry that we had above except that now $\vec{y}$ in (2.3) parametrizes an A_{k-1} singularity. It seems possible to deform a bit the singularity and scale the deformation parameter with R in such a way to retain a finite deformation in the limit. We will not study these limits in detail below but they are of clear physical interest.

3. Strings on pp-waves

It has been known for a while that strings on pp-wave NS backgrounds are exactly solvable [15]. The same is true for pp-waves on RR backgrounds. In fact, after we started thinking about this the paper by Metsaev [16] came out, so we will refer the reader to it for the details. The basic reason that strings on pp-waves are tractable is that the action dramatically simplifies in light cone gauge.

We start with the metric (2.4) and we choose light cone gauge $x^+ = \tau$ where τ is the worldsheet time. Then we see that the action for the eight transverse directions becomes just the action for eight massive bosons. Similarly the coupling to the RR background gives a mass for the eight transverse fermions.

So in light cone gauge we have eight massive bosons and fermions. It turns out that 16 of the 32 supesymmetries of the background are linearly realized in light cone gauge (just as in flat space). These sixteen supersymmetries commute with the light cone hamiltonian and so they imply that the bosons and fermions have the same mass, see [16] .

After the usual gauge fixing (see [16, 17]) the light cone action becomes

$$S = \frac{1}{2\pi\alpha'} \int dt \int_0^{2\pi\alpha' p^+} d\sigma \left[\frac{1}{2}\dot{z}^2 - \frac{1}{2}z'^2 - \frac{1}{2}\mu^2 z^2 + i\bar{S}(\not{\partial} + \mu I)S \right], \tag{3.1}$$

where $I = \Gamma^{1234}$ and S is a Majorana spinor on the worldsheet and a positive chirality SO(8) spinor under rotations in the eight transverse directions. We quantize this action by expanding all fields in Fourier modes on the circle labeled by σ. For each Fourier mode we get a harmonic oscillator (bosonic or fermionic depending on the field). Then the light cone hamiltonian is

$$2p^- = -p_+ = H_{lc} = \sum_{n=-\infty}^{+\infty} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} \,. \tag{3.2}$$

Here n is the label of the fourier mode, $n > 0$ label left movers and $n < 0$ right movers. N_n denotes the total occupation number of that mode, including bosons and fermions. Note that the ground state energy of bosonic oscillators is canceled by that of the fermionic oscillators.

In addition we have the condition that the total momentum on the string vanishes

$$P = \sum_{n=-\infty}^{\infty} n N_n = 0\,. \tag{3.3}$$

Note that for $n = 0$ we also have harmonic oscillators (as opposed to the situation in flat space). When only the $n = 0$ modes are excited we reproduce the spectrum of massless supergravity modes propagating on the plane wave geometry. A particle propagating on a plane wave geometry with fixed p^+ feels as if it was on a gravitational potential well, it cannot escape to infinity if its energy, p^-, is finite. Similarly a massless particle with zero p^+ can go to $r = \infty$ and back in finite x^- time (inversely proportional to μ). This is reminiscent to what happens for particles in AdS. In the limit that μ is very small, or in other words if

$$\mu\alpha' p^+ \ll 1 \tag{3.4}$$

we recover the flat space spectrum. Indeed we see from (2.3) that the metric reduces to the flat space metric if we set μ to zero.

It is also interesting to consider the opposite limit, where

$$\mu\alpha' p^+ \gg 1\,. \tag{3.5}$$

In this limit all the low lying string oscillator modes have almost the same energy. This limit (3.5) corresponds to a highly curved background with RR fields. In fact we will later see that the appearance of a large number of light modes is expected from the Yang-Mills theory.

It is useful to rewrite (3.2) in terms of the variables that are natural from the $AdS_5 \times S^5$ point of view. We find that the contribution to $\Delta - J = 2p^-$ of each oscillator is its frequency which can be written as

$$(\Delta - J)_n = w_n = \sqrt{1 + \frac{4\pi g N n^2}{J^2}} \tag{3.6}$$

using (2.5) and the fact that the AdS radius is given by $R^4 = 4\pi g N \alpha'^2$. Notice that N/J^2 remains fixed in the $N \to \infty$ limit that we are taking.

It is interesting to note that in the plane wave (2.4) we can also have giant gravitons as we have in $AdS_5 \times S^5$. These giants are D3-branes classically sitting at fixed x^- and wrapping the S^3 of the first four directions or the S^3 of the second four directions with a size

$$r^2 = 2\pi g p^+ \mu \alpha'^2\,, \tag{3.7}$$

where p^+ is the momentum carried by the giant graviton. This result follows in a straightforward fashion from the results in [18]. Its p^- eigenvalue is zero. We see that the description of these states in terms of D-branes is correct when their size is much bigger than the string scale. In terms of the Yang-Mills variables this happens when $J^2/N \gg 1/g$

There are many other interesting aspects of perturbative string propagation on plane waves that one could study. In appendix C we discuss the spectrum of strings on plane wave background of mixed NS and RR type. Note that for more general plane waves, for which the factor multiplying $(dx^+)^2$ is not quadratic, the action in light cone gauge is a more general interacting massive theory. We could have, for example, a Landau-Ginsburg theory. It would be nice to analyze these cases in detail. We can also have an x^+ dependent function, as discussed in [15].

It is well known that in conformal gauge the equation of motion for the background is conformal invariance of the two dimensional worldsheet theory. It would be nice to understand what the equation of motion for the background is in these more general massive cases, where we have chosen the light cone gauge fixing instead. In flat space conditions like $D = 26$ appear, in light cone gauge, from the proper realization of the non-linearly realized Lorentz generators. These plane wave backgrounds generically break those Lorentz generators.

4. Strings from $\mathcal{N} = 4$ Super Yang Mills

We are interested in the limit $N \to \infty$ where $g^2_{\rm YM}$ is kept fixed and small, $g^2_{\rm YM} \ll 1$. We want to consider states which carry parametrically large-R charge $J \sim \sqrt{N}$.[4] This R charge generator, J, is the SO(2) generator rotating two of the six scalar fields. We want to find the spectrum of states with $\Delta - J$ finite in this limit. We are interested in single trace states of the Yang Mills theory on $S^3 \times R$, or equivalently, the spectrum of dimensions of single trace operators of the theory on R^4. We will often go back and forth between the states and the corresponding operators.

Let us first start by understanding the operator with lowest value of $\Delta - J = 0$. There is a unique single trace operator with $\Delta - J = 0$, namely $\mathrm{Tr}[Z^J]$, where $Z \equiv \phi^5 + i\phi^6$ and the trace is over the N color indices. We are taking J to be the SO(2) generator rotating the plane 56. At weak coupling the dimension of this operator is J since each Z field has dimension one. This operator is a chiral primary and hence its dimension is protected by supersymmetry. It is associated to the vacuum state in light cone gauge, which is the unique state with zero light cone hamiltonian. In other words we have the correspondence

$$\frac{1}{\sqrt{J}N^{J/2}}\,\mathrm{Tr}[Z^J] \longleftrightarrow |0, p_+\rangle_{l.c.} \qquad (4.1)$$

We have normalized the operator as follows. When we compute $\langle \mathrm{Tr}[\bar{Z}^J](x)\,\mathrm{Tr}[Z^J](0)\rangle$ we have J possibilities for the contraction of the first $\bar{Z}$ but then planarity implies that we contract the second $\bar{Z}$ with a Z that is next to the first one we contracted and so on. Each of these contraction gives a factor of N. Normalizing this two point function to one we get the normalization factor in (4.1).[5]

[4] For reasons that we will discuss later we also need that $J/N^{1/2} \ll 1/g_{\rm YM}$. This latter condition comes from demanding that (3.7) is smaller than the string scale and it ensures that the states we consider are strings and not D-brane "giant gravitons" [18].

[5] In general in the free theory any contraction of a single trace operator with its complex conjugate one will give us a factor of N^n, where n is the number of fields appearing in the operator. If the number of fields is very large it is possible that non-planar contractions dominate over planar ones [19, 20] . In our case, due to the way we scale J this does not occur in the free theory.

Now we can consider other operators that we can build in the free theory. We can add other fields, or we can add derivatives of fields like $\partial_{(i_1} \cdots \partial_{i_n)} \phi^r$, where we only take the traceless combinations since the traces can be eliminated via the equations of motion. The order in which these operators are inserted in the trace is important. All operators are all "words" constructed by these fields up to the cyclic symmetry, these were discussed and counted in [2]. We will find it convenient to divide all fields, and derivatives of fields, that appear in the free theory according to their $\Delta - J$ eigenvalue. There is only one mode that has $\Delta - J = 0$, which is the mode used in (4.1). There are eight bosonic and eight fermionic modes with $\Delta - J = 1$. They arise as follows. First we have the four scalars in the directions not rotated by J, i.e. ϕ^i, $i = 1, 2, 3, 4$. Then we have derivatives of the field Z, $D_i Z = \partial_i Z + [A_i, Z]$, where $i = 1, 2, 3, 4$ are four directions in R^4. Finally there are eight fermionic operators $\chi^a_{J=1/2}$ which are the eight components with $J = 1/2$ of the sixteen component gaugino χ (the other eight components have $J = -1/2$). These eight components transform in the positive chirality spinor representation of $\mathrm{SO}(4) \times \mathrm{SO}(4)$.[6] We will focus first on operators built out of these fields and then we will discuss what happens when we include other fields, with $\Delta - J > 1$, such as $\bar{Z}$.

The state (4.1) describes a particular mode of ten dimensional supergravity in a particular wavefunction [5]. Let us now discuss how to generate all other massless supergravity modes. On the string theory side we construct all these states by applying the zero momentum oscillators a^i_0, $i = 1, \ldots, 8$ and S^b_0, $b = 1, \ldots 8$ on the light cone vacuum $|0, p_+\rangle_{l.c.}$. Since the modes on the string are massive all these zero momentum oscillators are harmonic oscillators, they all have the same light cone energy. So the total light cone energy is equal to the total number of oscillators that are acting on the light cone ground state. We know that in $AdS_5 \times S^5$ all gravity modes are in the same supermultiplet as the state of the form (4.1) [21]. The same is clearly true in the limit that we are considering. More precisely, the action of all supersymmetries and bosonic symmetries of the plane wave background (which are intimately related to the $AdS_5 \times S^5$ symmetries) generate all other ten dimensional massless modes with given p_+. For example, by acting by some of the rotations of S^5 that do not commute with the SO(2) symmetry that we singled out we create states of the form

$$\frac{1}{\sqrt{J}} \sum_l \frac{1}{\sqrt{J} N^{J/2+1/2}} \mathrm{Tr}[Z^l \phi^r Z^{J-l}] = \frac{1}{N^{J/2+1/2}} \mathrm{Tr}[\phi^r Z^J] \,, \tag{4.2}$$

where ϕ^r, $r = 1, 2, 3, 4$ is one of the scalars neutral under J. In (4.2) we used the cyclicity of the trace. Note that we have normalized the states appropriately in the planar limit. We can act any number of times by these generators and we get operators roughly of the form $\sum \mathrm{Tr}[\cdots z \phi^r z \cdots z \phi^k]$. where the sum is over all the possible orderings of the ϕs. We

[6] The first SO(4) corresponds to rotations in R^4, the space where the Yang Mills theory is defined, the second $\mathrm{SO}(4) \subset \mathrm{SO}(6)$ corresponds to rotations of the first four scalar fields, this is the subgroup of SO(6) that commutes with the SO(2), generated by J, that we singled out to perform the analysis. By positive chirality in $\mathrm{SO}(4) \times \mathrm{SO}(4)$ we mean that it has positive chirality under both SO(4)s or negative under both SO(4). Combining the spinor indices into SO(8), $\mathrm{SO}(4) \times \mathrm{SO}(4) \subset \mathrm{SO}(8)$ it has positive chirality under SO(8). Note that SO(8) is not a symmetry of the background.

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can repeat this discussion with the other $\Delta - J = 1$ fields. Each time we insert a new operator we sum over all possible locations where we can insert it. Here we are neglecting possible extra terms that we need when two $\Delta - J = 1$ fields are at the same position, these are subleading in a $1/J$ expansion and can be neglected in the large-J limit that we are considering. In other words, when we act with the symmetries that do not leave Z invariant we will change one of the Zs in (4.1) to a field with $\Delta - J = 1$, when we act again with one of the symmetries we can change one of the Zs that was left unchanged in the first step or we can act on the field that was already changed in the first step. This second possibility is of lower order in a $1/J$ expansion and we neglect it. We will always work in a "dilute gas" approximation where most of the fields in the operator are Zs and there are a few other fields sprinkled in the operator.

For example, a state with two excitations will be of the form

$$\sim \frac{1}{N^{J/2+1}} \frac{1}{\sqrt{J}} \sum_{l=1}^{J} \mathrm{Tr}\left[\phi^r Z^l \psi^b_{J=1/2} Z^{J-l}\right] , \tag{4.3}$$

where we used the cyclicity of the trace to put the ϕ^r operator at the beginning of the expression. We associate (4.3) to the string state $a_0^{\dagger k} S_0^{\dagger\ b} |0, p_+\rangle_{l.c.}$. Note that for planar diagrams it is very important to keep track of the position of the operators. For example, two operators of the form $\mathrm{Tr}[\phi^1 Z^l \phi^2 Z^{J-l}]$ with different values of l are orthogonal to each other in the planar limit (in the free theory).

The conclusion is that there is a precise correspondence between the supergravity modes and the operators. This is of course well known [4, 5, 6]. Indeed, we see from (3.2) that their $\Delta - J = 2p^-$ is indeed what we compute at weak coupling, as we expect from the BPS argument.

In order to understand non-supergravity modes in the bulk it is clear that what we need to understand the Yang Mills description of the states obtained by the action of the string oscillators which have $n \neq 0$. Let us consider first one of the string oscillators which creates a bosonic mode along one of the four directions that came from the S^5, let's say $a_n^{\dagger\ 8}$. We already understood that the action of $a_0^{\dagger\ 8}$ corresponds to insertions of an operator ϕ^4 on all possible positions along the "string of Z's". By a "string of Zs" we just mean a sequence of Z fields one next to the other such as we have in (4.1). We propose that $a_n^{\dagger 8}$ corresponds to the insertion of the same field ϕ^4 but now with a position dependent phase

$$\frac{1}{\sqrt{J}} \sum_{l=1}^{J} \frac{1}{\sqrt{J} N^{J/2+1/2}} \mathrm{Tr}[Z^l \phi^4 Z^{J-l}] e^{\frac{2\pi i n l}{J}} . \tag{4.4}$$

In fact the state (4.4) vanishes by cyclicity of the trace. This corresponds to the fact that we have the constraint that the total momentum along the string should vanish (3.3), so that we cannot insert only one $a_n^{\dagger\ i}$ oscillator. So we should insert more than one oscillator so that the total momentum is zero. For example we can consider the string state obtained by acting with the $a_n^{\dagger\ 8}$ and $a_{-n}^{\dagger\ 7}$, which has zero total momentum along the string. We

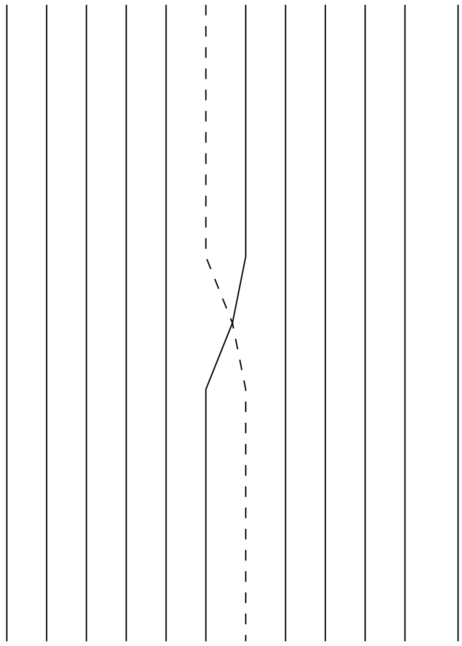


Figure 1: Diagrams that exchange the position of ϕ. They have "momentum", n, dependent contributions.

propose that this state should be identified with

$$a_n^{\dagger\,8} a_{-n}^{\dagger\,7}|0,p_+\rangle_{l.c.} \longleftrightarrow \frac{1}{\sqrt{J}} \sum_{l=1}^{J} \frac{1}{N^{J/2+1}} \mathrm{Tr}[\phi^3 Z^l \phi^4 Z^{J-l}] e^{\frac{2\pi i n l}{J}} \,, \tag{4.5}$$

where we used the cyclicity of the trace to simplify the expression. The general rule is pretty clear, for each oscillator mode along the string we associate one of the $\Delta - J = 1$ fields of the Yang-Mills theory and we sum over the insertion of this field at all possible positions with a phase proportional to the momentum. States whose total momentum is not zero along the string lead to operators that are automatically zero by cyclicity of the trace. In this way we enforce the $L_0 - \bar{L}_0 = 0$ constraint (3.3) on the string spectrum.

In summary, each string oscillator corresponds to the insertion of a $\Delta - J = 1$ field, summing over all positions with an n dependent phase, according to the rule

$$\begin{aligned} a^{\dagger i} &\longrightarrow D_i Z \qquad \text{for } i = 1, \ldots, 4 \\ a^{\dagger j} &\longrightarrow \phi^{j-4} \qquad \text{for } j = 5, \ldots, 8 \\ S^a &\longrightarrow \chi^a_{J=1/2} \,. \end{aligned} \tag{4.6}$$

In order to show that this identification makes sense we want to compute the conformal dimension, or more precisely $\Delta - J$, of these operators at large 't Hooft coupling and show that it matches (3.2). First note that if we set $gN/J^2 \sim 0$ in (3.6) we find that all modes, independently of n have the same energy, namely one. This is what we find at weak 't Hooft coupling where all operators of the form (4.5) have the same energy, independently of n. Expanding the string theory result (3.6) we find that the first correction is of the form

$$(\Delta - J)_n = w_n = 1 + \frac{2\pi g N n^2}{J^2} + \cdots . \tag{4.7}$$

This looks like a first order correction in the 't Hooft coupling and we can wonder if we can reproduce it by a a simple perturbative computation. Manipulations with non BPS operators suggest that anomalous dimensions grow like $g^2 N$ and that they disappear from the spectrum of the theory at strong coupling. However, this line of reasoning assumes that we keep the dimension of the operator in the free field theory (J in this case) fixed as we take the large-N limit. In our case the states we begin with are almost BPS; there are cancellations which depend on the free field theory dimension (J) which render the result finite even in the infinite 't Hooft coupling limit. The interesting diagrams arise from the following interaction vertex

$$\sim g_{\rm YM}^2 \,\mathrm{Tr}([Z,\phi^j][\bar{Z},\phi^j])\,. \tag{4.8}$$

This vertex leads to diagrams, such as shown in 1 which move the position of the ϕ^j operator along the "string" of Z's. In the free theory, once a ϕ^j operator is inserted at one position along the string it will stay there, states with ϕ^j's at different positions are orthogonal to each other in the planar limit (up to the cyclicity of the trace). We can think of the string of Zs in (4.1) as defining a lattice, when we insert an operator ϕ^1 at different positions along the string of Zs we are exciting an oscillator $b_l^\dagger$ at the site l on the lattice, $l = 1, \ldots, J$. The interaction term (4.8) can take an excitation from one site in the lattice to the neighboring site. So we see that the effects of (4.8) will be sensitive to the momentum n. In fact one can precisely reproduce (4.7) from (4.8) including the precise numerical coefficient. In appendix A we give the details of this computation.

Encouraged by the success of this comparison we want to reproduce the full square root[7] in (3.6). At first sight this seems a daunting computation since it involves an infinite number of corrections. These corrections nevertheless can be obtained from exponentiating (4.8) and taking into account that in (4.8) there are terms involving two creation operators $b^\dagger$ and two annihilation operators b. In other words we have $\phi \sim b + b^\dagger$. As we explained above, we can view ϕ's at different positions as different operators. So we introduce an operator $b_l^\dagger$ which introduces a ϕ operator at the site l along the string of Zs. Then the free hamiltonian plus the interaction term (4.8) can be thought of as

$$H \sim \sum_i b_l^\dagger b_l + \frac{g_{\rm YM}^2 N}{(2\pi)^2}\left[(b_l + b_l^\dagger) - (b_{l+1} + b_{l+1}^\dagger)\right]^2 . \tag{4.9}$$

In appendix A we give more details on the derivation of (4.9). In the large-N and -J limit it is clear that (4.9) reduces to the continuum hamiltonian

$$H = \int_0^L d\sigma \;\; \frac{1}{2}\left[\dot{\phi}^2 + \phi'^2 + \phi^2\right], \qquad L = J\sqrt{\frac{\pi}{gN}} \sim p^+ \tag{4.10}$$

which in turn is the correct expression for $H = p^- = \Delta - J$ for strings in the light cone gauge.

[7] Square roots of the 't Hooft coupling are ubiquitous in the AdS computations.

In summary, the "string of Zs" becomes the physical string and that each Z carries one unit of J which is one unit of p^+. Locality along the worldsheet of the string comes from the fact that planar diagrams allow onlycontractions of neighboring operators. So the Yang Mills theory gives a string bit model (see [22]), where each bit is a Z operator. Each bit carries one unit of J which through (4.10) is one unit of p^+.

The reader might, correctly, be thinking that all this seems too good to be true. In fact, we have neglected many other diagrams and many other operators which, at weak 't Hooft coupling also have small $\Delta - J$. In particular, we considered operators which arise by inserting the fields with $\Delta - J = 1$ but we did not consider the possibility of inserting fields corresponding to $\Delta - J = 2, 3, \ldots$, such as $\bar{Z}$, $\partial_k \phi^r$, $\partial_{(l}\partial_{k)} Z$, etc. The diagrams of the type we considered above would give rise to other 1+1 dimensional fields for each of these modes. These are present at weak 't Hooft coupling but they should not be present at strong coupling, since we do not see them in the string spectrum. We believe that what happens is that these fields get a large mass in the $N \to \infty$ limit. In other words, the operators get a large conformal dimension. In appendix A, we discuss the computation of the first correction to the energy (the conformal weight) of the of the state that results from inserting $\bar{Z}$ with some "momentum" n. In contrast to our previous computation for $\Delta - J = 1$ fields we find that besides an effective kinetic term as in (4.7) there is an n independent contribution that goes as gN with no extra powers of $1/J^2$. This is an indication that these excitations become very massive in the large-gN limit. In addition, we can compute the decay amplitude of $\bar{Z}$ into a pair of ϕ insertions. This is also very large, of order gN.

Though we have not done a similar computation for other fields with $\Delta - J > 1$, we believe that the same will be true for the other fields. In general we expect to find many terms in the effective lagrangian with coefficients that are of order gN with no inverse powers of J to suppress them. In other words, the lagrangian of Yang-Mills on S^3 acting on a state which contains a large number of Zs gives a lagrangian on a discretized spatial circle with an infinite number of KK modes. The coefficients of this effective lagrangian are factors of gN, so all fields will generically get very large masses.

The only fields that will not get a large mass are those whose mass is protected for some reason. The fields with $\Delta - J = 1$ correspond to Goldstone bosons and fermions of the symmetries broken by the state (4.1). Note that despite the fact that they morally are Goldstone bosons and fermions, their mass is non-zero, due to the fact that the symmetries that are broken do not commute with p^-, the light cone hamiltonian. The point is that their masses are determined, and hence protected, by the (super)symmetry algebra.

Having described how the single string Hilbert space arises it is natural to ask whether we can incorporate properly the string interactions. Clearly string interactions come when we include non-planar diagrams [1]. There are non-planar diagrams coming from the cubic vertex which are proportional to $g_{\rm YM}/N^{1/2}$. These go to zero in the large-N limit. There are also non-planar contributions that come from iterating the three point vertex or from the quartic vertex in the action. These are of order $g_{\rm YM}^2 \sim g$ compared to planar diagrams so that we get the right dependence on the string coupling g. In the discussion in this paragraph we have ignored the fact that J also becomes large in the limit we are considering. If we naively compute the factors of J that would appear we would seem to get a divergent contribution for the non-planar diagrams in this limit. Once we take into account that the cubic and quartic vertices contain commutators then the powers of J get reduced. From the gravity side we expect that some string interactions should become strong when $J/N^{1/2} \sim 1/g_{\rm YM}$. In other words, at these values of J we expect to find D-brane states in the gravity side, which means that the usual single trace description of operators is not valid any more, see discussion around (3.7). We have not been able to successfully reproduce this bound from the gauge theory side.

Some of the arguments used in this section look very reminiscent of the DLCQ description of matrix strings [23, 24]. It would be interesting to see if one can establish a connection between them. Notice that the DLCQ description of ten dimensional IIB theory is in terms of the M2-brane field theory. Since here we are extracting also a light cone description of IIB string theory we expect that there should be a direct connection.

It would also be nice to see if using any of these ideas we can get a better handle on other large-N Yang Mills theories, particularly non-supersymmetric ones. The mechanism by which strings appear in this paper is somewhat reminiscent of [25].

5. The matrix model for the DLCQ description of M-theory plane waves

In this section we point out that there is a nice, simple matrix model associated to these backgrounds. The M-theory pp-wave background is

$$\begin{aligned} ds^2 &= -4dx^- dx^+ - \left[\left(\frac{\mu}{3}\right)^2 (x_1^2 + x_2^2 + x_3^2) + \left(\frac{\mu}{6}\right)^2 (x_4^2 + \cdots + x_9^2) \right] dx^{+2} + d\vec{x}^2 \\ F_{+123} &= \mu \,. \end{aligned} \tag{5.1}$$

This metric arises as a limit similar to the one explained in section 2 for $AdS_4 \times S^7$ or $AdS_7 \times S^4$ (both cases give the same metric), see also [26].

This metric has a large symmetry group with 32 supersymmetries, the algebra is a contraction of the $AdS_{4,7} \times S^{7,4}$ superalgebras as expected from the fact that they are limits of the $AdS_{4,7} \times S^{7,4}$ superalgebras. In analogy to the discussion [27]–[30] we do DLCQ along the direction $x^- \sim x^- + 2\pi R$, and we consider the sector of the theory with momentum $2p^+ = -p_- = N/R$. Then the dynamics of the theory in this sector is given by the U(N) matrix model

$$\begin{aligned} S &= S_0 + S_{\rm mass} \\ S_0 &= \int dt\, {\rm Tr}\Bigg[\sum_{j=1}^{9} \frac{1}{2(2R)} (D_0\phi^j)^2 + \Psi^T D_0 \Psi + \frac{(2R)}{4} \sum_{j,k=1}^{9} [\phi^j,\phi^k]^2 + \\ &\qquad + \sum_{j=1}^{9} i(2R)(\Psi^T \gamma^i [\Psi,\phi^j]) \Bigg] \\ S_{\rm mass} &= \int dt\, {\rm Tr}\Bigg[\frac{1}{2(2R)} \left(-\left(\frac{\mu}{3}\right)^2 \sum_{j=1,2,3} (\phi^j)^2 - \left(\frac{\mu}{6}\right)^2 \sum_{j=4}^{9} (\phi^j)^2 \right) - \frac{\mu}{4} \Psi^T \gamma_{123} \Psi - \\ &\qquad - \frac{\mu}{3} i \sum_{j,k,l=1}^{3} {\rm Tr}(\phi^j \phi^k \phi^l) \epsilon_{jkl} \Bigg] \,, \end{aligned} \tag{5.2}$$

where we have set l_p=1. We also have that $t = x^+$ and $\phi = \frac{r}{2\pi}$ where r is the physical distance in eleven dimensions. S_0 is the usual matrix theory of [27].[8] $S_{\rm mass}$ adds mass to the scalar fields and fermion fields, plus a term associated to the Myers effect [31].

The action (5.2) has the transformation rules

$$\begin{aligned}
\delta\phi^i &= \Psi^T \gamma^i \epsilon(t) \\
\delta\Psi &= \left(\frac{1}{(2R)} D_0 \phi^i \gamma^i + \frac{\mu}{6(2R)} \sum_{i=1}^{3} \phi^i \gamma^i \gamma_{123} - \frac{\mu}{3(2R)} \sum_{i=4}^{9} \phi^i \gamma^i \gamma_{123} + \frac{i}{2} [\phi^i, \phi^j] \gamma_{ij} \right) \epsilon(t) \\
\delta A_0 &= \Psi^T \epsilon(t) \\
\epsilon(t) &= e^{-\frac{\mu}{12} \gamma_{123} t} \epsilon_0 \,.
\end{aligned} \tag{5.3}$$

In appendix B we show that the action (5.2) is determined by the supersymmetry algebra of the plane wave metric [12]. The matrix model hamiltonian, associated to this action is equal to $H = -p_+$.

Note that the bosons and fermions have different masses, three of the bosons have mass $\mu/3$ while six of them have mass $\mu/6$. On the other hand all the fermions have mass $\mu/4$. This is possible because the supersymmetries (5.3) are time dependent and therefore do not commute with the hamiltonian. This is in agreement with the susy algebra of plane waves [12], see appendix B. It is easy to check that the vacuum energy is still zero. This is good since there is a state with zero p_+ which corresponds to a single type of graviton mode.

Let us look at the fully supersymmetric solutions of this action. Imposing that $\delta\Psi = 0$ we find that the only solutions are

$$[\phi^i, \phi^j] = i \frac{\mu}{6R} \epsilon_{ijk} \phi^k \quad i,j,k = 1,2,3 \quad \dot{\phi}^i = 0 \ \text{ for all } i \ \text{ and } \ \phi^i = 0, \quad i = 4, \dots, 9 \tag{5.4}$$

that is, a fuzzy sphere in the 1,2,3 directions of physical radius

$$r \sim 2\pi \sqrt{\frac{{\rm Tr}[\sum_i {\phi^i}^2]}{N}} \sim \pi \frac{\mu}{6} \frac{N}{R} \,. \tag{5.5}$$

We see that the mass terms remove completely the moduli space and leave only a discrete set of solutions, after modding out by gauge transformations. This is convenient, as the structure of the ground states is governed by the semiclassical approximation. One does not need to solve the full quantum mechanical problem of the ground state wave function, an issue which frequently arises in the more standard matrix model [27] and that has proved very difficult to approach.

The solutions are labelled by all possible ways of dividing an N dimensional representation of SU(2) into irreducible representations. This number is equal to the number of partitions of N, which is also the number of multiple graviton states with $p_+ = 0$, $-p_- = N/R$ in a naive fock space description.

[8] To compare with [27] note that due to the form of the metric and the way we define R, $2R_{our} = R_{BFSS}$. We normalize l_p so that $\sqrt{\alpha'} = l_p g^{-1/3}$ when we go to the IIA theory.

The solutions (5.4) are related to "giant gravitons" in the plane wave background (5.1) which are M2-branes wrapping the S^2 given by $\sum_{i=1}^{3} x_i^2$=constant and classically sitting at a fixed position x^-, but with nonzero momentum p_- (but zero light cone energy $-p_+ = 0$). The supergravity computation of the radius, similar to that in [18] gives again (5.5).

The plane wave geometry also admits giant gravitons which are M5-branes wrapping the S^5 given by $\sum_{i=4}^{9} x_i^2$ =constant. We can similarly compute the value of the radius from the supergravity side and we get

$$r^4 = \frac{8\pi^2}{3}\mu(-p_-)\,. \tag{5.6}$$

This does not appear as a classical solution of the matrix model (5.2), which is of course not unrelated to the difficulty of seeing the M5-brane in the matrix mode [32]. It is interesting to notice however that if we write the two sphere radius (5.5) in terms of the coupling constant of the matrix model we find $\hat{\phi} \sim \mu/g$, where the action with an overall factor of $\frac{1}{g^2}$. This scaling of $\hat{\phi}$ is precisely what we expect for a classical solution. On the other hand, if we express (5.6) in the same way we obtain $\hat{\phi}^4 \sim 1/g$. This scaling with g does not correspond to a classical solution of (5.2) and therefore it is natural that we do not find it. The situation seems similar to the one encountered in the analysis of vacua of mass deformed $\mathcal{N} = 4$ Yang Mills done by Polchinski and Strassler [33]. They find that the process of D3-branes blowing up into D5-branes can be described classically in the Yang-Mills theory, while the process of D3-branes blowing up into NS 5-branes requires that one takes into account the quantum effects. It is therefore natural to conjecture that the vacuum with $x^i = 0$ in the quantum mechanics theory corresponds to a single large M5-brane.

It is interesting to note that there are other solutions that preserve a fraction of the supersymmetry and that are time dependent. These are commuting configurations of the type

$$\begin{aligned} (\phi^4 + i\phi^5)(t) &= e^{-i\frac{\mu}{6}t}(\phi^4 + i\phi^5)(0) \\ [\phi^i(0), \phi^j(0)] &= 0, \qquad i = 4, 5 \\ \gamma_{12345}\epsilon_0 &= \epsilon_0 \end{aligned} \tag{5.7}$$

and similar ones obtained by replacing 4,5 with any other pair of indices out of 4,. . . ,9, as well as a similar solution with a pair of indices from $1, 2, 3$ with exponent $e^{-i\frac{\mu}{3}t}$.

There are many other interesting questions regarding plane waves, such as the precise nature of the observables, etc. They also seem to admit a holographic description, since as we remarked above plane waves have much in common with AdS. We plan to continue investigating these questions.

Acknowledgments

We would like to thank R. Dijkgraaf, I. Klebanov N. Seiberg, H. Verlinde and E. Witten for discussions.

This research was supported in part by DOE grants DE-FGO2-91ER40654 and DE-FG02-90ER40542.

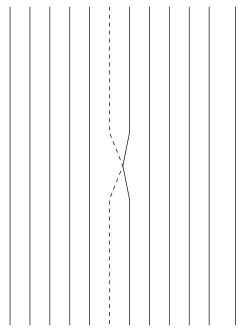


Figure 2: Diagrams that have "momentum", n, independent contributions.

Figure 3: Example of diagrams that have "momentum" independent contributions that we do not compute directly. These diagrams are the same if we replace $\phi \to Z$.

A. More detailed computations

In subsection A.1 of this appendix we describe in more detail the computation of the numerical coefficient in (4.7). In subsection A.2 we discuss how to exponentiate those corrections to obtain (3.6). Finally, in the third subsection we explain how some $\Delta - J{=}2$ excitations get a large mass and decay rapidly to $\Delta - J{=}1$ excitations.

A.1 Computation of the first perturbative correction

In this subsection we discuss the computation of the first perturbative correction to the anomalous dimension of an operator of the form (4.5). To compute we analyze the correlation function of two such operators. We consider operators containing a large number, J, of Zs with a few ϕs distributed along the "string of Zs". In other words, we sum over all possible insertion points of each field ϕ with a phase of the form $e^{i2\pi nj/J}$ where j is the position of ϕ along the "string of Zs". We are interested in perturbative corrections to the dimension of the operator coming from the vertex (4.8). Since the ϕs are few and far apart we can consider each insertion of ϕ independently, up to $1/J$ corrections.

The terms in the (euclidean) Yang-Mills action that we will be interested in are normalized as

$$S = \frac{1}{2\pi g}\int d^4x\,\mathrm{Tr}\left(\frac{1}{2}(D\phi^I)^2 - \frac{1}{4}\sum_{IJ}[\phi^I,\phi^J]^2\right), \tag{A.1}$$

where I, J run over six values. We wrote the square of the Yang-Mills coupling in terms of what in AdS is the string coupling that transforms as $g \to 1/g$ under S-duality. The trace is just the usual trace of an $N \times N$ matrix. We define $Z = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6)$. Then the propagators are

$$\langle Z_i{}^j(x)\bar{Z}_k{}^l(0)\rangle = \langle \phi_i{}^j(x)\bar{\phi}_k{}^l(0)\rangle = \delta_i^l\delta_k^j\,\frac{2\pi g}{4\pi^2}\,\frac{1}{|x|^2}\,. \tag{A.2}$$

In (A.1) there is an interaction term of the form the form $\frac{1}{2\pi g}\int d^4x\,\mathrm{Tr}([Z,\phi][\bar{Z},\phi])$, where ϕ is one of the transverse scalars, let's say $\phi=\phi^1$. We focus first on the diagrams that give a contribution that depends on the "momentum" n. These arise from interactions that shift the position of ϕ in the operator, such as the ones shown in 1. These interactions come from a particular ordering of the commutator term in the action, $\frac{1}{2\pi g}\int d^4x 2\,\mathrm{Tr}[\phi Z\phi\bar{Z}]$. These contributions give

$$\langle O(x)O^*(0)\rangle = \frac{\mathcal{N}}{|x|^{2\Delta}}\left[1+N(2\pi g)4\cos\frac{2\pi n}{J}I(x)\right], \tag{A.3}$$

where $\mathcal{N}$ is a normalization factor and $I(x)$ is the integral

$$I(x) = \frac{|x|^4}{(4\pi^2)^2}\int d^4y\frac{1}{y^4(x-y)^4}\sim\frac{1}{4\pi^2}\log|x|\Lambda+\text{finite}\,. \tag{A.4}$$

We extracted the log divergent piece of the integral since it is the one that reflects the change in the conformal dimension of the operator.

In addition to the diagrams we considered above there are other diagrams, such as the ones shown in 2 and 3, which do not depend on n. We know that for $n=0$ the sum of all diagrams cancels since in that case we have a protected operator and there is no change in the conformal dimension. In other words, including the n independent diagrams amounts to replacing the cosine in (A.3) by

$$\cos\frac{2\pi n}{J}-1\,. \tag{A.5}$$

In conclusion we find that for large-J and -N the first correction to the ϕ contribution to the correlator is

$$\langle O(x)O^*(0)\rangle = \frac{\mathcal{N}}{|x|^{2\Delta}}\left[1-\frac{4\pi gNn^2}{J^2}\log(|x|\Lambda)\right] \tag{A.6}$$

which implies that the contribution of the operator ϕ inserted in the "string of Zs" with momentum n gives a contribution to the anomalous dimension

$$(\Delta-J)_n = 1+\frac{2\pi gNn^2}{J^2}\,. \tag{A.7}$$

There are similar computations we could do for insertions of D_iZ or the fermions $\chi^a_{J=1/2}$. In the case of the fermions the important interaction term will be a Yukawa coupling of the form $\bar{\chi}\Gamma_z[Z\chi]+\bar{\chi}\Gamma_{\bar{z}}[\bar{Z},\chi]$. We have not done these computations explicitly since the 16 supersymmetries preserved by the state (4.1) relate them to the computation we did above for the insertion of a ϕ operator.

The full square root arises from iterating these diagrams. This will be more transparent in the formalism we discuss in the next subsection.

A.2 A hamiltonian description

In this subsection we reformulate the results of the previous subsection in a hamiltonian formalism and we explain why we get a relativistic action on the string once we iterate the particular interaction that we are considering.

Here we will consider the Yang-Mills theory defined on $S^3 \times R$. All fields of the theory can be expanded in KK harmonics on S^3. States of this theory are in one to one correspondence with local operators on R^4. We take the radius of S^3 equal to one so that the energy of the state is equal to the conformal dimension of the corresponding operator. For weak coupling, $g_{\rm YM}^2 N \ll 1$, the scalar fields give rise to a KK tower. The lowest energy state is the constant mode on S^3. Due to the curvature coupling there is effectively a mass term for the scalar fields with a mass equal to one (when the radius of S^3 is one). So the constant mode on S^3 is described by a harmonic oscillator of frequency equal to one. Due to the color indices we have N^2 harmonic oscillators with commutation relations

$$[a_j{}^i, (a^\dagger)_k{}^l] = \delta_j^l \delta_k^i \tag{A.8}$$

for each mode. The fields ϕ^5, ϕ^6 lead to oscillators which can be combined into a pair of oscillators a_+ and a_- with definite J charge. From now one we denote by a the a_+ oscillator. The operator (4.1) corresponds to the state

$$\frac{1}{\sqrt{J} N^{J/2}} \,\mathrm{Tr}[a^J]|0\rangle\,. \tag{A.9}$$

This is a single trace state. We will be interested only in single trace states. In the large-N limit multiple trace states are orthogonal to single trace states in the free theory.[9] In the free theory we can build all states by forming all possible "words" out of all the oscillators associated to all the KK modes of all the fields. The order is important up to cyclicity of the trace. When we perform inner products or contractions of states we will restrict only to planar contractions. Those are efficiently reproduced by replacing the standard oscillators $a_{\alpha}{}^i{}_j$, by Cuntz oscillators a_α where α labels the type of field and the KK mode [35, 36]. The Cuntz algebra is

$$a_\alpha a_\beta^\dagger = \delta_{\alpha,\beta} \tag{A.10}$$

and no other relation other than the one coming from the completeness relation

$$\sum_\alpha a_\alpha^\dagger a_\alpha = 1 - |0\rangle\langle 0|\,. \tag{A.11}$$

More precisely, in order to take into account the factors of N we replace $\frac{a_{\alpha}{}^i{}_j}{\sqrt{N}} \to a_\alpha$ where the latter is a Cuntz oscillator. This algebra is rather useful for keeping track of the planarity of the contractions but one needs to be careful about enforcing properly the

[9] This might not be true even in the free theory if J is too large [19, 20] but for our case where $J \sim N^{1/2}$ it is indeed true. In the interacting theory we expect, from the gravity side, non-planar corrections when $J/\sqrt{N} \sim \frac{1}{g_{\rm YM}}$.

cyclicity of the trace, etc. As emphasized in [36] this algebra is a useful framework to study large-N matrix theories. In our case we will be interested in states of the form

$$\cdots a^\dagger b^\dagger a^\dagger \cdots a^\dagger b^\dagger a^\dagger \cdots |0\rangle\,, \tag{A.12}$$

where the dots indicate a sequence of $a^\dagger$ operators. We will be interested in the action of the gauge theory hamiltonian on such states where we have a small number of $b^\dagger$. We will be interested in the interaction term in the hamiltonian of the form

$$g_{\rm YM}^2\,{\rm Tr}([Z,\phi][\bar{Z}\phi]) \to g_{\rm YM}^2 N[a^\dagger,\phi][a,\phi]\,, \tag{A.13}$$

where in the second term we think of $\phi \sim b+b^\dagger$ where ϕ is one of the transverse scalar fields and b is the corresponding Cuntz oscillator. We neglect self contractions in the hamiltonian since those will be canceled by other propagator corrections in the case of $\mathcal{N}=4$ SYM. In the left hand side of the interaction term there are many possible oscillators in the fields Z, $\bar{Z}$, we have only keep the piece involving the oscillator with $\Delta - J = 0$. An interaction amounts to an insertion of the hamiltonian (A.13) in any position of the state (A.12). We also need to sum in the right hand side of (A.13) over all possible orderings. Since there is a large number of $a^\dagger$ in the state (A.12) we an define b_j oscillators which are the b oscillators inserted at the jth position along the string. In this way the effective hamiltonian reduces to

$$H = \sum_j b_j^\dagger b_j + \frac{gN}{2\pi}\left(b_j + b_j^\dagger - b_{j+1} - b_{j+1}^\dagger\right)^2\,, \tag{A.14}$$

where again, as in the computation of the last subsection, the term proportional to $b_i^\dagger b_i$ was obtained indirectly by appealing to the BPS property of the state in the case of zero momentum excitations. We can see from this effective hamiltonian that the first correction on a state of the form $\sum_l e^{\frac{i2\pi nl}{J}} b_l^\dagger|0\rangle'$ is indeed (4.7). Here the vacuum $|0\rangle'$ for the b_l oscillators is really the state (A.9). The effective hamiltonian is then essentially the discretized hamiltonian of a massive scalar field in 1+1 dimension, where we discretize only the space direction, except that the oscillators in (A.14) have the usual commutation relations for different sites but they are Cuntz oscillators on the same site. We can see, however, that if we define the oscillators

$$b_n^\dagger \equiv \frac{1}{\sqrt{J}}\sum_{l=1}^{J} e^{\frac{i2\pi ln}{J}} b_l^\dagger \tag{A.15}$$

then the b_n oscillators obey the standard commutation relations up to terms of order $1/J$ which we neglect in the large-J limit. For this reason the large-J limit of (A.14) will give the same as the continuum hamiltonian

$$H = \int_0^L d\sigma \frac{1}{2}\left[\dot{\phi}^2 + \phi'^2 + \phi^2\right], \qquad L = J\sqrt{\frac{\pi}{gN}}\,. \tag{A.16}$$

In 4 we see see the form of the diagrams that we are summing to obtain (A.16). Note that when we diagonalize the new hamiltonian (A.14) the new vacuum will be related to the old vacuum by a Bogoliubov transformation, so that in a sense there will be a fair number of $b^\dagger$s in the new vacuum. Supersymmetry ensures that the vacuum energy does not change, so that we still have $\Delta - J = 0$ for the new vacuum.

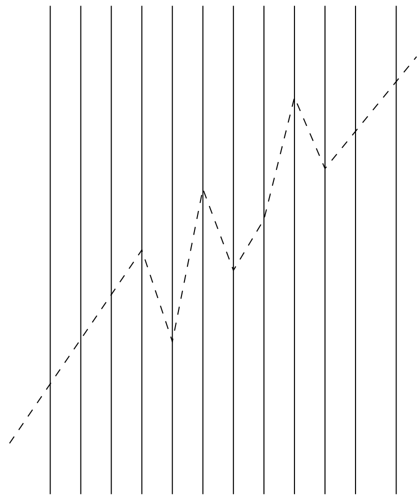


Figure 4: This is a schematic representation of the diagrams that we are resumming to obtain (A.16). The doted line is the ϕ propagator. Each crossing of a Z line is an interaction.

A.3 The fate of the other fields

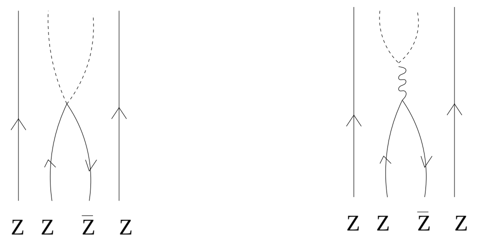
Let us now understand what happens when we insert in (4.1) a field with $\Delta - J > 1$. We will study the simplest case which arises when we insert the field $\bar{Z}$. This field has $\Delta - J = 2$. We can insert this field with arbitrary "momentum" n in the operator (as long as we make sure that (3.3) is obeyed). We will now show that the correction to its dimension now does not vanish for zero momentum. We consider an operator of the form

$$\sum_l e^{\frac{i2\pi ln}{J}} \,\mathrm{Tr}[\cdots ZZ\bar{Z}Z\cdots] \,, \tag{A.17}$$

where l indicates the position of $\bar{Z}$ along the "string of Zs" and the dots indicate a large number of Z fields together with possibly other insertions of other fields, etc. Since we work in the dilute gas approximation, where J is very large, we can consider $\bar{Z}$ in isolation from other insertions of other operators. We can now compute the first order correction, in gN, to the anomalous dimension of (A.17). The relevant diagrams come from a vertex of the form $\frac{-1}{2\pi g}\int d^4x \frac{1}{2}\,\mathrm{Tr}([Z,\bar{Z}])^2$. The computation of these diagrams is identical to the one done in the first subsection of this appendix, the only difference comes when we consider the combinatoric factors in the diagram. There are again other diagrams (similar to those in 3) involving the exchange of gauge fields, corrections to the propagator, etc, which we can effectively compute by noticing that if we change, for $n = 0$, $\bar{Z} \to Z$ in (A.17), then we have a BPS state and all diagrams should cancel. Putting this all together we obtain

$$(\Delta - J)_n = 2 + \frac{gN}{4\pi}\left(4 + \frac{4\pi^2 n^2}{J^2}\right) + \cdots \,, \tag{A.18}$$

where we expanded the result in powers of $1/J$. In contrast to (A.7) we now find a contribution that is not finite in the $N \to \infty$ limit that we are taking. We have computed the correction only to first order in gN and we are extrapolating to $gN \to \infty$. This is



(a) (b)

Figure 5: (a) This diagram contributes to the decay of $\bar{Z}$ into two ϕs. (b) This diagram is zero.

not justified. So the above computation should be taken as an indication that insertions of $\bar{Z}$ do not lead to finite energy excitations in the effective 1+1 dimensional theory in the large-N limit that we are taking.

As we explained above, we expect that for $\Delta - J = 1$ the fields do not get a large mass because they are Goldstone bosons or fermions of the symmetries broken by (4.1).

We can similarly compute the decay amplitude, to first order in gN, of the excitation with $\Delta - J = 2$, created by the insertion of $\bar{Z}$, into excitations with $\Delta - J = 1$ created by a pair of insertions of the transverse scalars $\phi^r \phi^r$, $r = 1, 2, 3, 4$. These are given by the diagrams such as the one shown in figure 5a. Again we find a result proportional to gN with no powers of $1/J^2$ to suppress it.

In summary, we expect that in the large-N limit all excitations created by fields with $\Delta - J > 1$ become very massive and rapidly decay to excitations with $\Delta - J = 1$.

B. Supersymmetry of the massive matrix model

B.1 Symmetry algebra

In this subsection we make some remarks about the super symmetry algebra. We will consider the 11d wave [12] but similar remarks apply for the 10 dimensional waves [7].

We define a generator $e = -p_-$ and a generator $h = -p_+$. The generator e commutes with all the other operators. Some of the (anti)commutation relations are

$$
\begin{aligned}
[a_i, a_j^\dagger] &= e\delta_{ij}\,, & i,j &= 1,\ldots,9 \\
[h, a_i^\dagger] &= \frac{\mu}{3} a_i^\dagger\,, & [h, a_i] &= -\frac{\mu}{3} a_i \quad i = 1,2,3 \\
[h, a_i^\dagger] &= \frac{\mu}{6} a_i^\dagger\,, & [h, a_i] &= -\frac{\mu}{6} a_i\,, \quad i,j = 4,\ldots,9 \\
\{b_{\alpha\dot\beta}, b^{\dagger\;\gamma\dot\delta}\} &= e\delta_\alpha^\gamma \delta_{\dot\beta}^{\dot\delta} & 0 &= \{b, b'\} = \{b^\dagger, b'^\dagger\} = [b,a] = [b,a^\dagger] \\
[h, b^{\dagger\;\gamma\dot\delta}] &= \frac{\mu}{4} b^{\dagger\;\gamma\dot\delta}\,, & [h, b_{\alpha\dot\beta}] &= -\frac{\mu}{4} b_{\alpha\dot\beta} \\
[h, Q_{\alpha\dot\beta}] &= \frac{\mu}{12} Q_{\alpha\dot\beta}\,, & [h, S^{\gamma\dot\delta}] &= -\frac{\mu}{12} S^{\gamma\dot\delta} \\
\{Q, Q\} &= \{S, S\} = 0\,, & S_{\alpha\dot\beta}^\dagger &= Q_{\alpha\dot\beta}
\end{aligned}
$$

$$
\{Q_{\alpha\dot\beta}, S^{\gamma\dot\delta}\} = \delta_\alpha^\gamma \delta_{\dot\beta}^{\dot\delta} h + i\frac{\mu}{6}\delta_{\dot\beta}^{\dot\delta} \sum_{i,j\le 3} (\gamma_{ij})_\alpha^\gamma M_{ij} + i\frac{\mu}{12}\delta_\alpha^\gamma \sum_{i,j\ge 4} (\Gamma_{ij})_{\dot\beta}^{\dot\delta} M_{ij}\,, \tag{B.1}
$$

where the undoted greek indices indicate spinor indices of SU(2) an the doted ones denote spinor indices of SO(6) (the ones downstairs are in the **4** of SU(4) and the upstairs one are in the $\bar{\mathbf{4}}$ of SU(4)) and γ^i and Γ^j are three and six dimensional gamma matrices respectively.[10] In addition we have (anti)commutators of the S and Q with bs or as which give as or bs. We will not write those since we will give them implicitly below when we discuss the superparticle. The main observation we want to make is that the structure of the representations of this algebra is very simple. Since e commutes with everything we can diagonalize it. Then the commutation relations of the as and bs (and their adjoints) become bosonic and fermionic harmonic oscillators. Then the rest of the symmetries acts linearly on these oscillators. We can identify h with the lightcone hamiltonian, so we see that the $a^\dagger$ and $b^\dagger$ oscillators describe the center of mass motion of the state. In fact we could subtract from Q, S, h, M_{ij} an expression bilinear in these oscillators (which is a realization of Q, S, h, M_{ij} in terms of oscillators) so than then Q, S, h, M_{ij} act on the relative state. Note that Q, S are supersymmetries that do not commute with the hamiltonian.

In the matrix model, the oscillators a and b are going to result from quantizing the U(1) degree of freedom and the shift of Q, S, h, M_{ij} that we mentioned above amounts to separating the U(1) degree of freedom to leave the SU(N) degrees of freedom.

B.2 Plane wave limit of the 10d IIB $AdS_5 \times S_5$ action

Here we prove that the GS action of Metsaev [16] can be obtained as a limit of the $AdS_5 \times S_5$ action of [34].

There is a general formalism one can use in both cases. Indeed, as shown in [37], for D-branes propagating in supercoset manifolds, one can write down an action in terms of supervielbeins (vielbeins of the target superspace realized as a coset manifold). The kinetic term is always of the type

$$S = \int_M d^n\sigma \sqrt{g} g^{ij} L_i^A L_j^A \,, \tag{B.2}$$

where L_i^A are the bosonic components of the supervielbein 1-forms pulled back on the worldsheet. In general there can be also a WZ term, defined as the integral of a form on a n+1 dimensional manifold with M as boundary.

The supervielbeins are found from the general procedure in [37] as

$$L^A = L_0^A + 2\theta^\alpha f^A_{\alpha\beta} \left(\frac{\sinh^2 M/2}{M^2}\right)^\beta_\gamma (D\theta)^\gamma \,, \tag{B.3}$$

and where the matrix M is defined by

$$(M^2)^\alpha_\beta = -\theta^\gamma f^\alpha_{\gamma A} \theta^\delta f^\beta_{\delta A} \tag{B.4}$$

the coefficients $f^A_{\alpha\beta}$ are the structure constants of the fermi-fermi part of the superalgebra

[10] The relation of the generators in (B.1) and those in [12] is schematically as follows $a^j \sim e^j + ie^{*j}$ and similarly for $a^\dagger$, the bs and $b^\dagger$s are linear combinations of Q_+ in [12] and similarly S and Q are linear combinations of Q_- in [12].

$\{F_\alpha, F_\beta\} = f^A_{\alpha\beta} B_A$. If θ is constant, one gets the WZ parametrization of superspace. Here

$$(D\theta)^\alpha = d\theta^\alpha + (L^A_0 B_A \theta)^\alpha \tag{B.5}$$

is the Killing spinor operator acting on the θ's (the Killing spinor equation would be $D\epsilon(x) = 0$).

The GS string action in a general supergravity background was given in [38] and is

$$S = -\frac{1}{2}\int_{\partial M_3} d^2\sigma \sqrt{g} g^{ij} L_i^{\hat{a}} L_j^{\hat{a}} + i \int_{M_3} s^{IJ} L^{\hat{a}} \wedge \bar{L}^I \gamma^{\hat{a}} \wedge L^J \,, \tag{B.6}$$

where $L^{\hat{a}}$ are the bosonic supervielbeins and L^I the fermionic ones.

In the case of $AdS_5 \times S_5$ the simple form of the action based on the above approach has been found in [16, 34]:

$$S = -\frac{1}{2}\int d^2\sigma \left(\sqrt{-g} g^{ij} L_i^{\hat{a}} L_j^{\hat{a}} + 4i\epsilon^{ij} \int_0^1 ds s^{IJ} L_{is}^{\hat{a}} \bar{\Theta}^I \Gamma^{\hat{a}} L_{is}^J \right), \tag{B.7}$$

where

$$\begin{aligned} L_s^I &= \left(\frac{\sinh(sM)}{M} D\Theta \right)^I \\ L_s^{\hat{a}} &= e^{\hat{a}}_{\hat{m}} dX^{\hat{m}} - 4i\bar{\Theta}^I \Gamma^{\hat{a}} \left(\frac{\sinh^2(sM/2)}{M^2} D\Theta \right)^I . \end{aligned} \tag{B.8}$$

The fermionic light-cone gauge was fixed in [41], and is the same as in flat space, namely $\Gamma^+\theta = 0$. With this fermionic light-cone gauge, one gets that the matrix $M^2 = 0$, and so the only nontrivial information is encoded in $D\Theta$. But that has the general form

$$D\Theta^I = \left(\delta^{IJ} \left(d + \frac{1}{4}\omega^{\mu\nu}\gamma^{\mu\nu} \right) + \frac{i}{48} e^\mu F_{\mu\mu_1\dots\mu_4} \Gamma^{\mu_1\dots\mu_4} \epsilon^{IJ} \right) \Theta^J \tag{B.9}$$

and consequently it has the correct limit from the $AdS_5 \times S_5$ case to the pp wave case. The last step is the fixing of the bosonic light-cone gauge, which for the $AdS_5 \times S_5$ case was done in [39]. Metsaev [16], using the gauge

$$\sqrt{g} g^{ab} = \eta^{ab} \qquad x^+(\tau, \sigma) = \tau \tag{B.10}$$

finds then the action

$$L = -\frac{1}{2}\partial_a x^I \partial^a x^I - \frac{\mu^2}{2} x_I^2 - i\bar{\psi}\bar{\gamma}^- \rho^a \partial_a \psi + i\mu\bar{\psi}\bar{\gamma}^- \Pi\psi \,. \tag{B.11}$$

B.3 Matrix theory action

The action for a single D0-brane can be obtained as the superparticle action moving in (5.1) in the Green-Schwarz formulation. Indeed, for a D0-brane in flat space, the light-cone gauge superparticle action gives the free massless bosons X^i and fermions θ (spinors of SO(9)), which is the free D0 action.

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As we mentioned in the case of the GS string, the super-brane action has a kinetic and a WZ term. But in the case of the superparticle, there is no 2d form one can write down (except for the target space $AdS_2 \times S_2$ where one has the target space invariants ϵ_{ab}). So the superparticle action has only the kinetic term.

The supervielbeins for the 11d supersymmetric pp-wave can be obtained as a limit from the $AdS_7 \times S_4$ supervielbeins, just as above for the 10d wave as a limit of the $AdS_5 \times S_5$. Indeed, from the above formalism, the supervielbeins can be written in a universal form depending only on the structure constants $f^A_{\alpha\beta}$ of the superalgebra, and in terms of the Killing spinor operator. But we know that the wave space symmetry algebras are a contraction of the $AdS \times S$ ones, and that the Killing spinor operators are also a similar limit (they only depend on F).

The supervielbeins for the $AdS_7 \times S_4$ case have been given in [42]. If one takes the general formulas there and substitutes $F_{+123} = \mu$ and the fact that ω^{-i} are the only nonzero components of $\omega^{\hat{\mu}\hat{\nu}}$one obtains

$$D\theta = d\theta + \frac{\mu}{12}\left(e^{\hat{r}}\Gamma_{\hat{r}}{}^{+123} - 8e^{[+}\Gamma^{123]}\right)\theta - \frac{1}{2}\omega^{-i}\Gamma_{-i}\theta \tag{B.12}$$

and also

$$\begin{aligned}(M^2)^{\alpha}_{\beta} = & \frac{\mu}{6}[(\Gamma_{\hat{r}}{}^{+123} - 8\delta^{[+}_{\hat{r}}\Gamma^{123]})\theta]^{\alpha}(\bar{\theta}\Gamma^{\hat{r}})_{\beta} - \\ & -\frac{\mu}{12}[(\Gamma_{\hat{r}\hat{s}}\theta)^{\alpha}(\bar{\theta}\Gamma^{\hat{r}\hat{s}+123})_{\beta} + 24(\Gamma_{[-1}\theta)^{\alpha}(\bar{\theta}\Gamma_{23]})_{\beta}]\,.\end{aligned} \tag{B.13}$$

The superparticle action

$$\int dte^{-1}L^A_t L^A_t \tag{B.14}$$

will have a k symmetry similar to the one of the free superparticle with $L^{\mu}_t = \dot{x}^{\mu} - i\bar{\theta}^A\Gamma^{\mu}\dot{\theta}^A$. This k symmetry needs to be gauge fixed by choosing the fermionic light-cone gauge. The procedure is exactly similar to the superstring in $AdS_5 \times S_5$ and its limit the 10 d wave (see [16]). As there, one can choose the gauge

$$\Gamma^+\theta = 0 \qquad \bar{\theta}\Gamma^+ = 0 \tag{B.15}$$

which we can see from the expression of the $AdS_7 \times S_4$ M^2 above that makes $M^2 = 0$, and so

$$L^A = dx^{\hat{\mu}}e^A_{\hat{\mu}} + \frac{1}{2}\bar{\theta}\Gamma^A D\theta \tag{B.16}$$

and where

$$D\theta = d\theta + \frac{\mu}{12}(e^{\hat{r}}\Gamma_{\hat{r}}{}^{+123} - 8e^{[+}\Gamma^{123]})\theta - \frac{1}{2}\omega^{-i}\Gamma_{-i}\theta = d\theta + \frac{\mu}{12}e^+\Gamma^{-+}\Gamma^{123}\theta - \frac{\mu}{6}e^+\Gamma^{123}\theta\,, \tag{B.17}$$

where we have used the gauge condition to kill the terms with Γ^- and Γ_{+i}. One can then see that we get (in spacetime light cone parametrization)

$$L^+ = e^+ = dx^+ \qquad L^i = e^i = dx^i \tag{B.18}$$

and

$$L^- = e^- + \frac{1}{2}\bar{\theta}\Gamma^- D\theta\,; \quad e^- = dx^- - \frac{1}{2}\left(\frac{\mu}{3}\right)^2 \sum_{i=1,2,3} (x^i)^2 dx^+ - \frac{1}{2}\left(\frac{\mu}{6}\right)^2 \sum_{i=4}^{9} (x^i)^2 dx^+ \quad \text{(B.19)}$$

to be used in the action

$$S = \int dt \left(2L_t^+ L_t^- + L_t^i L_t^i\right). \quad \text{(B.20)}$$

Then fixing the bosonic light cone gauge $e = 1, x^-(t) = t$ one gets the action

$$S = \int dt \left[(\dot{X}^i)^2 - \left(\frac{\mu}{3}\right)^2 \sum_{i=1,2,3} (X^i)^2 - \left(\frac{\mu}{6}\right)^2 \sum_{i=4}^{9} (X^i)^2 + \bar{\theta}\Gamma^- \dot{\theta} - \frac{\mu}{4}\bar{\theta}\Gamma^- \Gamma^{123}\theta\right]. \quad \text{(B.21)}$$

We now rewrite the 11d fermions and gamma matrices in terms of 9d ones. We choose the representation

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \sigma_3 \\ \Gamma^0 &= 1 \otimes i\sigma_2 \\ \Gamma^{11} &= 1 \otimes \sigma_1\,. \end{aligned} \quad \text{(B.22)}$$

And we also choose a real (Majorana) representation for the spinors and gamma matrices: $C = \Gamma_0, \bar{\theta} = \theta^T C = \theta^\dagger \Gamma_0$. Then we have

$$\Gamma^- = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \Gamma^+ = \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \Gamma_0\Gamma^- = -\sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \Gamma^{+-} = 1 \otimes \sigma_3\,. \quad \text{(B.23)}$$

Then, take

$$\theta = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{so} \quad \Gamma^+ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \Rightarrow \psi_2 = 0\,. \quad \text{(B.24)}$$

So, take

$$\theta = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \quad \text{(B.25)}$$

and so the fermion terms in the action sum up to

$$\sqrt{2}\left(\psi^T \dot{\psi} + \frac{\mu}{4}\psi^T \gamma^{123}\psi\right). \quad \text{(B.26)}$$

We can now absorb the $\sqrt{2}$ in front of this expression in the definition of the fermions.

We turn to proving susy of this action, and generalizing it to the nonabelian case. We will leave the coefficient of the fermion mass term free, since we will find another solution for it in the abelian case.

Let us then start with the lagrangian

$$L = \sum_{i=1}^{9} (\dot{X}^i)^2 - \left(\frac{\mu}{3}\right)^2 \sum_{i=1,2,3} (X^i)^2 - \left(\frac{\mu}{6}\right)^2 \sum_{i=4}^{9} (X^i)^2 + \Psi^T \dot{\Psi} - a\left(\frac{\mu}{4}\right) \Psi^T \gamma_{123} \Psi \quad \text{(B.27)}$$

and look for a susy transformation of the type

$$\begin{aligned}
\delta X^i &= \Psi^T \gamma^i \epsilon(t) \\
\delta\Psi &= \dot{X}^i \gamma^i \epsilon(t) + \mu X^i \gamma^i M_i' \epsilon(t) \\
\epsilon(t) &= e^{\mu M t} \epsilon_0 \,.
\end{aligned} \tag{B.28}$$

Then the terms of order 1 in the susy transformation cancel , the terms of order μ give the equation

$$M_i' = \pm \frac{a}{4\gamma_{123}} - M \,, \tag{B.29}$$

where the two values are for i=1,2,3 and i=4,...,9 respectively, and the terms of order μ^2 give

$$\begin{aligned}
M_i' M - \frac{1}{9} - \frac{a}{4\gamma_{123}} M_i' &= 0 \qquad i = 1,2,3 \\
M_i' M - \frac{1}{36} + \frac{a}{4\gamma_{123}} M_i' &= 0 \qquad i = 4,\ldots,9 \,.
\end{aligned} \tag{B.30}$$

We then obtain

$$\begin{aligned}
M &= b\gamma_{123} \\
M_i' &= \left(\pm\frac{a}{4} - b\right)\gamma_{123}
\end{aligned} \tag{B.31}$$

and $a = 1$ or $1/3$ (2 solutions) and $b = -1/12$ or $-1/4$ (the 2 corresponding solutions). There are also solutions where we change the sign of both a and b, but these correspond to the symmetry $\mu \to -\mu$.

The extension to the nonabelian theory is obvious; besides the usual commutator terms which are present in the lagrangian and susy rules in flat space, we have an extra coupling of order μ. Indeed, Myers [31] has found a term $F_{tijk}\,\mathrm{Tr}(X^i X^j X^k)$ in the action for N D0-branes in constant RR field. In our case, after the limit to the plane wave geometry (infinite boost), the coupling is

$$F_{+ijk}\,\mathrm{Tr}(X^i X^j X^k) \sim \mu\,\mathrm{Tr}(X^i X^j X^k)\epsilon_{ijk} \,. \tag{B.32}$$

So the lagrangian is

$$\begin{aligned}
L = &\sum_{i=1}^{9} (\dot{X}^i)^2 - \left(\frac{\mu}{3}\right)^2 \sum_{i=1,2,3} (X^i)^2 - \left(\frac{\mu}{6}\right)^2 \sum_{i=4}^{9} (X^i)^2 + \Psi^T\dot{\Psi} - a\left(\frac{\mu}{4}\right)\Psi^T\gamma_{123}\Psi + \\
&+ d\mu g \sum_{i,j,k=1}^{3} \mathrm{Tr}(X^i X^j X^k)\epsilon_{ijk} + 2g^2\,\mathrm{Tr}([X^i, X^j]^2) + 2ig\,\mathrm{Tr}(\Psi^T\gamma^i[\Psi, X^i]) \,.
\end{aligned} \tag{B.33}$$

And the susy rules are

$$\begin{aligned}
\delta X^i &= \Psi^T \gamma^i \epsilon(t) \\
\delta\Psi &= \left(\dot{X}^i\gamma^i + \mu X^i \gamma^i \left(\pm\frac{a}{4} - b\right)\gamma_{123} + ig[X^i, X^j]\gamma_{ij}\right)\epsilon(t) \\
\epsilon(t) &= e^{\mu M t}\epsilon_0 \,.
\end{aligned} \tag{B.34}$$

The terms of order g^0 in the susy transformation of L work the same way as for one D0-brane, since they are bilinear in fields. The terms of order g cancel (they would fix the coefficient of the $\Psi\psi X$ term in the action). The terms of order μg are proportional to $\mathrm{Tr}(\Psi^T\gamma^{ij}\gamma_{123}\epsilon(t)[X^i,X^j])$ and split into i, j both $=4,\ldots,9$, one of i, $j=1,2,3$ and the other $=4,\ldots,9$ which both give the equation

$$3b+\frac{a}{4}=0 \tag{B.35}$$

and the case when both i, j are 1,2,3 which gives

$$d=2\left(b-\frac{a}{4}\right). \tag{B.36}$$

So now a and b are restricted to just $a=1$, $b=-1/12$. This solution is the one we found from the general formalism. The terms of order g^2 cancel (they would fix the coefficient of the $[X,X]^2$ term in the action).

The action has the almost the same nonlinearly realized susy as in flat space. In flat space, the nonlinear susy is $\delta\Psi=\epsilon$ (constant), and the X's constant. In our case, we have

$$\delta\Psi=\epsilon(t)=e^{\mu\frac{a}{4}\gamma_{123}t}\epsilon_0\,. \tag{B.37}$$

C. Strings on mixed NS and RR plane waves

As we remarked above, we can consider the limit of section 2, for the $AdS_3\times S^3$ backgrounds. It is interesting to consider such a limit in a situation where we have a mixture of NS and RR three form field strength. The six dimensional plane wave metric that we obtain has the form

$$\begin{aligned}
ds^2 &= -4dx^+dx^- -\mu^2\vec{y}^{\,2}(dx^+)^2+d\vec{y}^{\,2}\\
H^{NS}_{+12} &= H^{NS}_{+34}=C_1\mu\cos\alpha\\
H^{RR}_{+12} &= H^{RR}_{+34}=C_2\mu\sin\alpha\,,
\end{aligned} \tag{C.1}$$

where $\vec{y}$ parametrizes a point on R^4 and α is a fixed parameter which allows us to interpolate between the purely NS background $\alpha=0$ and the purely RR background $\alpha=\pi/2$. The constants C_1,C_2 depend on the string coupling and the normalization of the RR and NS field strengths. In addition to the six coordinates in (2.1) we have four additional directions which we can take to be a T^4 (or a K3).

The light cone action becomes

$$\begin{aligned}
S=\frac{1}{2\pi\alpha'}\int dt\int_0^{2\pi\alpha'p^+}d\sigma\frac{1}{2}\left[|\dot{Z}_i|^2-|Z_i'+i\cos\alpha Z_i|^2-\sin^2\alpha\mu^2|Z_i|^2\right]+\\
+\bar{S}(\sigma^0\partial_0+\sigma^1(\partial_1+\cos\alpha I)+\sin\alpha\mu I)S+\mathcal{L}_{T^4}\,,
\end{aligned} \tag{C.2}$$

where x denotes the four coordinates of T^4 and $I\equiv\Gamma^{12}$. We have also defined $Z_1=y_1+iy_2$ and $Z_2=y_3+iy_4$. The fermions S in (C.2) have positive chirality in the directions 1234 (and hence also positive chirality in the T^4 directions). The lagrangian $\mathcal{L}_{T^4}$ includes the modes living on T^4 as well as for the fermions that have negative chirality on the directions 1234 which are still massless. Only half of the fermions get a mass.

The light cone hamiltonian is then

$$\begin{aligned} p^+ = H_{lc} = \sum_{n=-\infty}^{\infty} N_n \sqrt{\sin^2\alpha^2\mu^2 + \left(\cos\alpha\mu + \frac{n}{\alpha' p^+}\right)^2} + \\ + 2\frac{L_0^{T^4} + \bar{L}_0^{T^4}}{\alpha' p_+}\,, \end{aligned} \tag{C.3}$$

where the first line takes into account the massive bosons and fermions and the second line takes into account the T^4 bosons and the four massless fermions. We also have the condition that the total momentum along the string is zero.

We see that in the pure RR case we get something quite similar to the previous result. For the pure NS case the spectrum can be viewed as arising from twisted boundary conditions along the string. In that case, when $\alpha' p^+\mu = n$ we have a new zero mode appearing. When we excite this zero mode we obtain a string that winds n times around the origin and has zero light cone energy due to the cancellation of the gravitational and "electric" energy. These are analogous to the long strings much discussed in AdS_3 with NS background [40]. As soon as $\cos\alpha \neq 1$ these new zero modes disappear, as is expected.

It would be nice to see if it is possible to reproduce the spectrum (C.3) from the dual CFT of the D1-D5 system.

References

[1] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[2] A.M. Polyakov, *String theory and quark confinement*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 1 [hep-th/9711002]; *Gauge fields and space-time*, hep-th/0110196.

[3] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[5] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[6] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[7] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242].

[8] R. Penrose, *Any spacetime has a plane wave as a limit*, in *Differential geometry and relativity*, Reidel, Dordrecht 1976, pp. 271–275.

[9] K. Sfetsos, *Gauging a nonsemisimple WZW model*, *Phys. Lett.* **B 324** (1994) 335 [hep-th/9311010];
K. Sfetsos and A.A. Tseytlin, *Four-dimensional plane wave string solutions with coset CFT description*, *Nucl. Phys.* **B 427** (1994) 245 [hep-th/9404063].

JHEP04(2002)013

[10] R. Gueven, *Plane waves in effective field theories of superstrings*, *Phys. Lett.* **B 191** (1987) 275; *Plane wave limits and T-duality*, *Phys. Lett.* **B 482** (2000) 255.

[11] J. Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, *Phys. Lett.* **B 134** (1984) 194.

[12] J. Figueroa-O'Farrill and G. Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, *J. High Energy Phys.* **08** (2001) 036 [hep-th/0105308].

[13] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[14] S. Kachru and E. Silverstein, *4d conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183].

[15] See for example: D. Amati and C. Klimčík, *Strings in a shock wave background and generation of curved geometry from flat space string theory*, *Phys. Lett.* **B 210** (1988) 92;
G.T. Horowitz and A.R. Steif, *Space-time singularities in string theory*, *Phys. Rev. Lett.* **64** (1990) 260;
H.J. de Vega and N. Sanchez, *Quantum string propagation through gravitational shock waves*, *Phys. Lett.* **B 244** (1990) 215;
H.J. de Vega and N. Sanchez, *Space-time singularities in string theory and string propagation through gravitational shock waves*, *Phys. Rev. Lett.* **65** (1990) 1517;
O. Jofre and C. Nuñez, *Strings in plane wave backgrounds revisited*, *Phys. Rev.* **D 50** (1994) 5232 [hep-th/9311187].

[16] R.R. Metsaev, *Type-IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044].

[17] J. Polchinski, *String theory*, Cambridge, University Press, Cambridge 1998.

[18] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *J. High Energy Phys.* **06** (2000) 008 [hep-th/0003075].

[19] V. Balasubramanian, M. Berkooz, A. Naqvi and M.J. Strassler, *Giant gravitons in conformal field theory*, hep-th/0107119.

[20] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, hep-th/0111222.

[21] M. Günaydin and N. Marcus, *The spectrum of the S^5 compactification of the chiral $N = 2$, $D = 10$ supergravity and the unitary supermultiplets of* $\mathrm{U}(2, 2/4)$, *Class. and Quant. Grav.* **2** (1985) L11;
H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *The mass spectrum of chiral $N = 2$ $D = 10$ supergravity on S^5*, *Phys. Rev.* **D 32** (1985) 389.

[22] R. Giles and C.B. Thorn, *A lattice approach to string theory*, *Phys. Rev.* **D 16** (1977) 366;
C.B. Thorn, *Reformulating string theory with the $1/N$ expansion*, hep-th/9405069.

[23] L. Motl, *Proposals on nonperturbative superstring interactions*, hep-th/9701025.

[24] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix string theory*, *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030].

[25] I.R. Klebanov and L. Susskind, *Continuum strings from discrete field theories*, *Nucl. Phys.* **B 309** (1988) 175.

[26] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, hep-th/0201081.

JHEP04(2002)013

[27] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[28] L. Susskind, *Another conjecture about M(atrix) theory*, hep-th/9704080.

[29] A. Sen, *D0-branes on T_n and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 51 [hep-th/9709220].

[30] N. Seiberg, *Why is the matrix model correct?*, *Phys. Rev. Lett.* **79** (1997) 3577 [hep-th/9710009].

[31] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[32] T. Banks, N. Seiberg and S.H. Shenker, *Branes from matrices*, *Nucl. Phys.* **B 490** (1997) 91 [hep-th/9612157].

[33] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[34] R. Kallosh and A.A. Tseytlin, *Simplifying superstring action on $AdS_5 \times S^5$*, *J. High Energy Phys.* **10** (1998) 016 [hep-th/9808088].

[35] D.V. Voiculescu, K.J. Dykema and A. Nica *Free random variables*, AMS, Providence 1992.

[36] R. Gopakumar and D.J. Gross, *Mastering the master field*, *Nucl. Phys.* **B 451** (1995) 379 [hep-th/9411021].

[37] R. Kallosh, J. Rahmfeld and A. Rajaraman, *Near horizon superspace*, *J. High Energy Phys.* **09** (1998) 002 [hep-th/9805217].

[38] M.T. Grisaru, P. Howe, L. Mezincescu, B. Nilsson and P.K. Townsend, *N=2 superstrings in a supergravity background*, *Phys. Lett.* **B 162** (1985) 116.

[39] R.R. Metsaev, C.B. Thorn and A.A. Tseytlin, *Light-cone superstring in AdS space-time*, *Nucl. Phys.* **B 596** (2001) 151 [hep-th/0009171].

[40] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT*, *J. High Energy Phys.* **04** (1999) 017 [hep-th/9903224];
J. Maldacena and H. Ooguri, *Strings in AdS_3 and $\mathrm{SL}(2,\mathbb{R})$ WZW model, 1*, *J. Math. Phys.* **42** (2001) 2929 [hep-th/0001053].

[41] R.R. Metsaev and A.A. Tseytlin, *Superstring action in $AdS_5 \times S^5$: kappa-symmetry light cone gauge*, *Phys. Rev.* **D 63** (2001) 046002 [hep-th/0007036].

[42] B. de Wit, K. Peeters, J. Plefka and A. Sevrin, *The M-theory two-brane in $AdS_4 \times S_7$ and $AdS_7 \times S_4$*, *Phys. Lett.* **B 443** (1998) 153 [hep-th/9808052].

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